Porządkowe średnie ważone OWA i WOWA jako kryteria oceny w warunkach niepewności

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The Ordered Weighted Average

- $\Theta = (\theta_1, \theta_2, \ldots, \theta_m)$ ordering map (nonincreasing order)
  $\theta_1(\mathbf{y})$ – the largest outcome
  $\theta_2(\mathbf{y})$ – the second largest, $\theta_3(\mathbf{y})$ – the third largest, etc.

- Ordered Weighted Aggregation [Yager,88]
  weights w_i assigned to ordered outcomes: $A_{\mathbf{w}}(\mathbf{y}) = \sum_{i=1}^{m} w_i \theta_i(\mathbf{y})$

- Special cases

$w_1 = w_2 = \ldots = w_m = 1/m$ — arithmetic mean

$w_1 = 1$, $w_2 = \ldots = w_m = 0$ — maximum

$w_1 = \ldots = w_{m-1} = 0$, $w_m = 1$ — minimum

$w_k = 1$, $w_i = 0$ for $i \neq k$ — $\frac{m-k+1}{m}$–quantile

The Ordered Weighted Average

- orness (andnes) measure

$$\text{orness}(\mathbf{w}) = \sum_{i=1}^{m} \frac{m-i}{m-1} w_i, \qquad \text{andness}(\mathbf{w}) = 1 - \text{orness}(\mathbf{w})$$

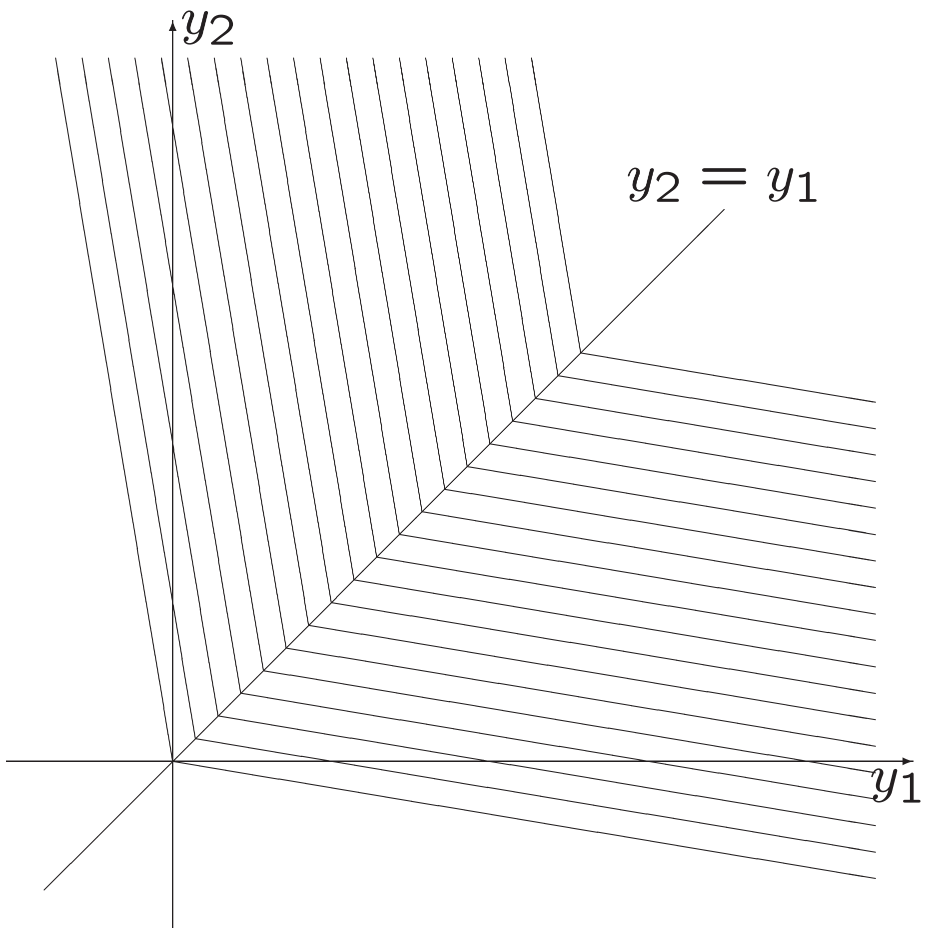
  max aggregation (fuzzy 'or') with $\mathbf{w} = (1, 0, \ldots, 0)$ – orness$(\mathbf{w}) = 1$
  min aggregation (fuzzy 'and') with $\mathbf{w} = (0, \ldots, 0, 1)$ – orness$(\mathbf{w}) = 0$
  the average (arithmetic mean) – orness$((1/m, 1/m, \ldots, 1/m)) = 1/2$

- $0 < w_1 \leq w_2 \leq \ldots \leq w_m$ – totally and-like aggregation
  $w_1 \geq w_2 \geq \ldots \geq w_m > 0$ – totally or-like aggregation
  in the sense that they remain valid for any subsequences ($2 \leq k \leq m$)

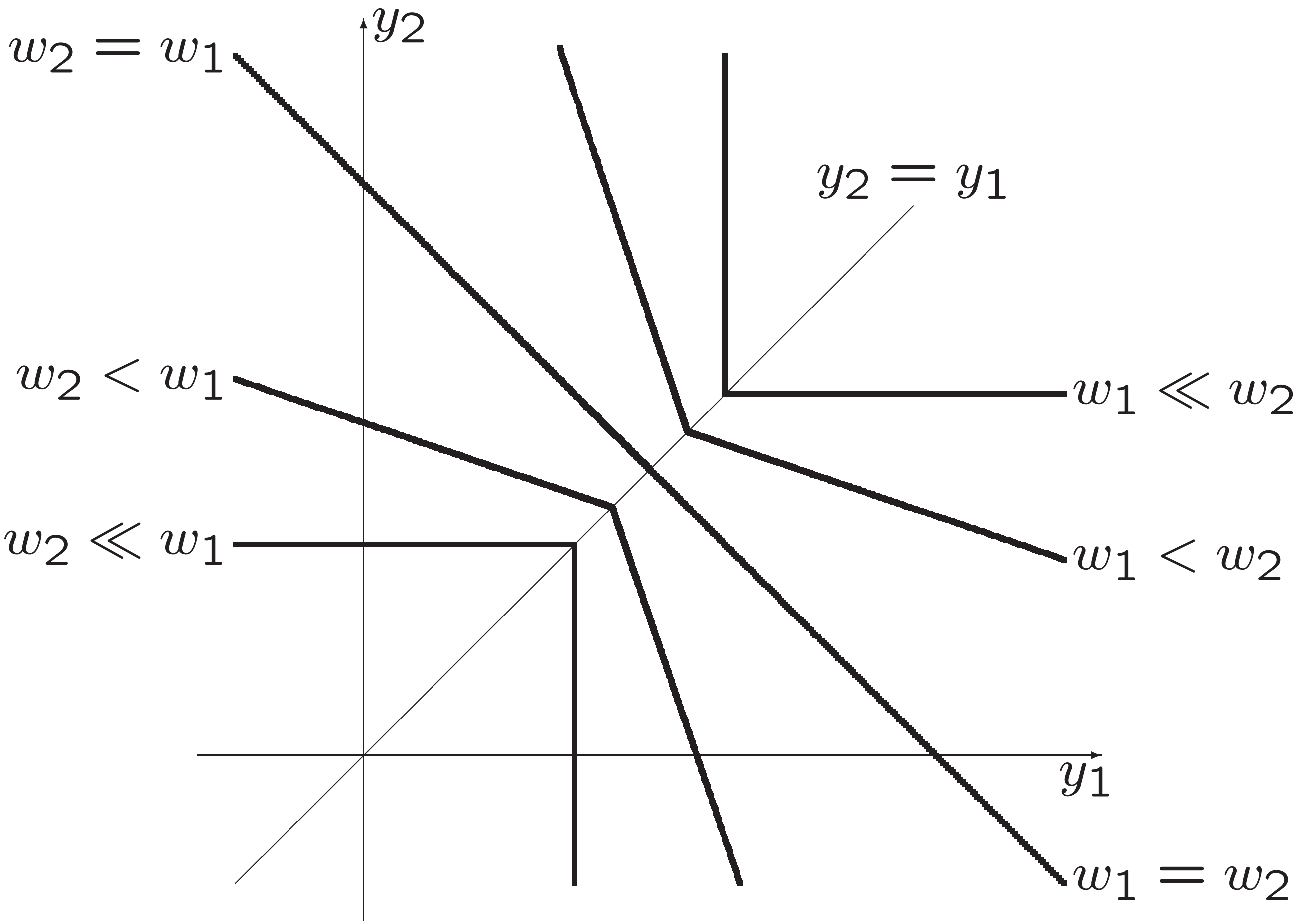
$$\sum_{j=1}^{k} \frac{k-j}{k-1} w_{i_j} \geq \frac{1}{2} \quad \text{or} \quad \sum_{j=1}^{k} \frac{k-j}{k-1} w_{i_j} \leq \frac{1}{2}$$

  Weights monotonicity is necessary to achieve the above total orness and andness properties.

OWA – Sample isoline contours for $w_1 \leq w_2$



Various isoline contours for OWA



Decisions under Uncertainty

- Decision variables: x_j, $j = 1, \ldots, n$

- (Scalar) Optimization: $\max\{f(\mathbf{x}) \ : \ \mathbf{x} \in Q\}$

- Nondeterministic outcome

  – Decisions under uncertainty:
  Scenarios $S_i, \quad i = 1, \ldots, m$, Outcome realizations $y_i = f_i(\mathbf{x})$

  $$\max\{(f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})) \ : \ \mathbf{x} \in Q\}$$

  – Decisions under risk: scenario probabilities p_i

Decisions under Risk

- Equally probable scenarios S_i

$$\max\{(f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})) \; : \; \mathbf{x} \in Q\}$$

$\mathbf{y} = (y_1, y_2, \ldots, y_m)$ decision outcome in a form of a lottery with m equally probable ($p_i = \frac{1}{m}$) tickets y_i.

- Impartiality (symmetry)

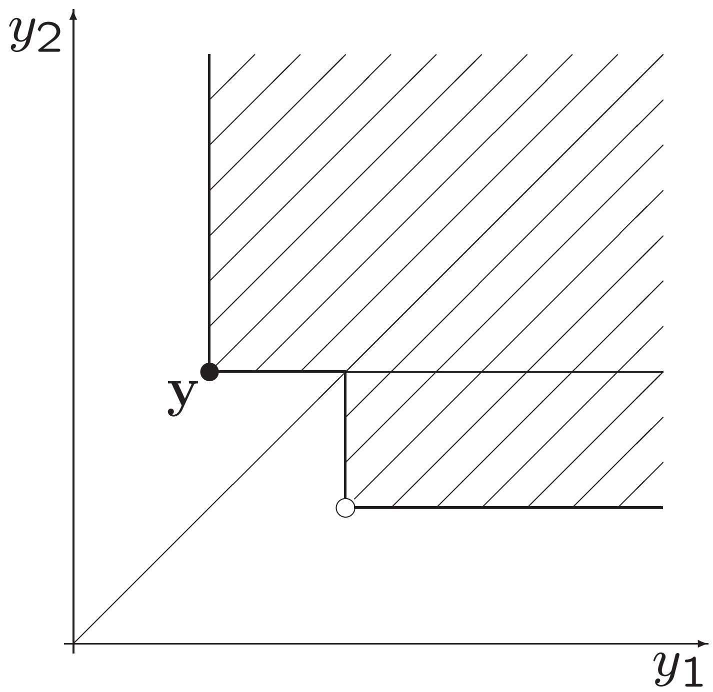
$$(y_{\tau(1)}, y_{\tau(2)}, \ldots, y_{\tau(m)}) \cong (y_1, y_2, \ldots, y_m)$$

- Risk aversion

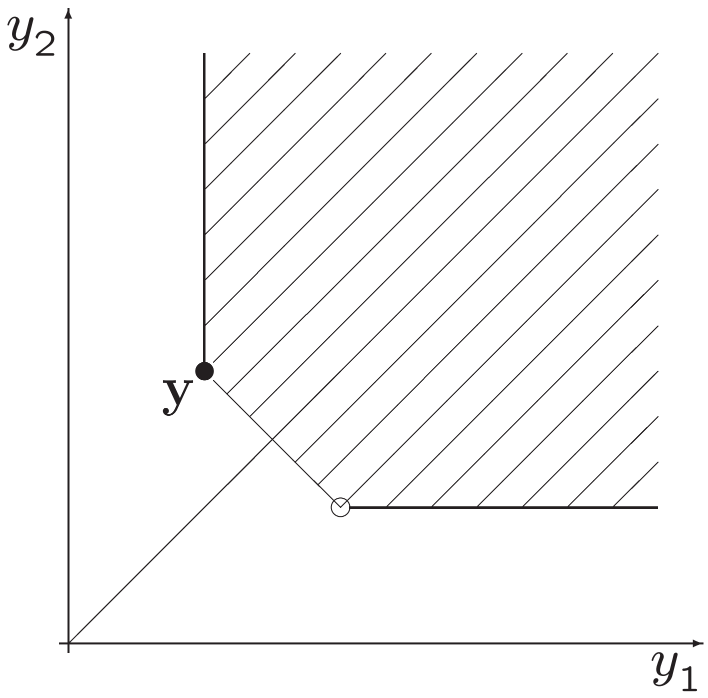
$$y_{i'} > y_{i''} \quad \Rightarrow \quad \mathbf{y} - \varepsilon \mathbf{e}_{i'} + \varepsilon \mathbf{e}_{i''} \succ \mathbf{y}$$

strict preference of an equitable trans fer

Symmetric dominance structure (general preferences)



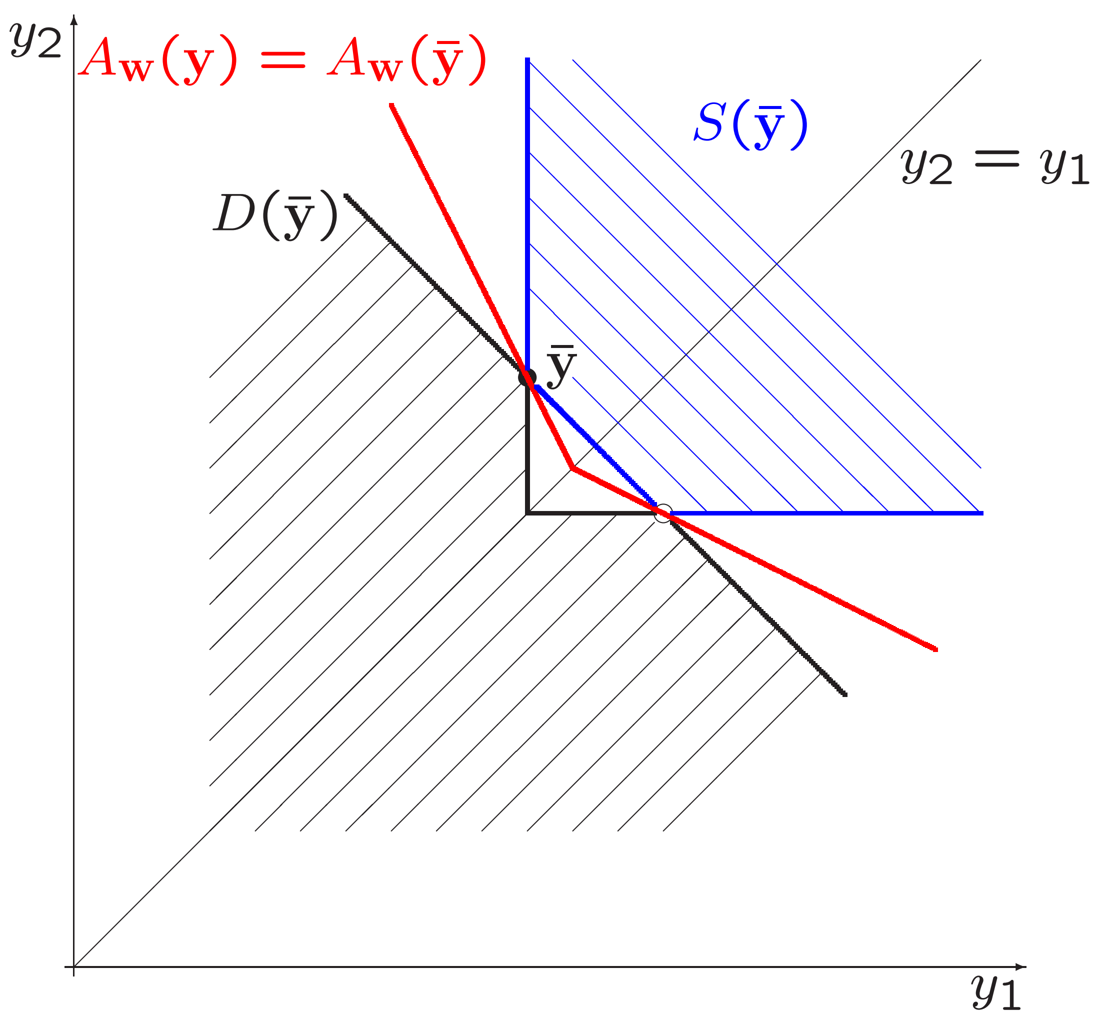
Equitable dominance structure
(risk aversion)



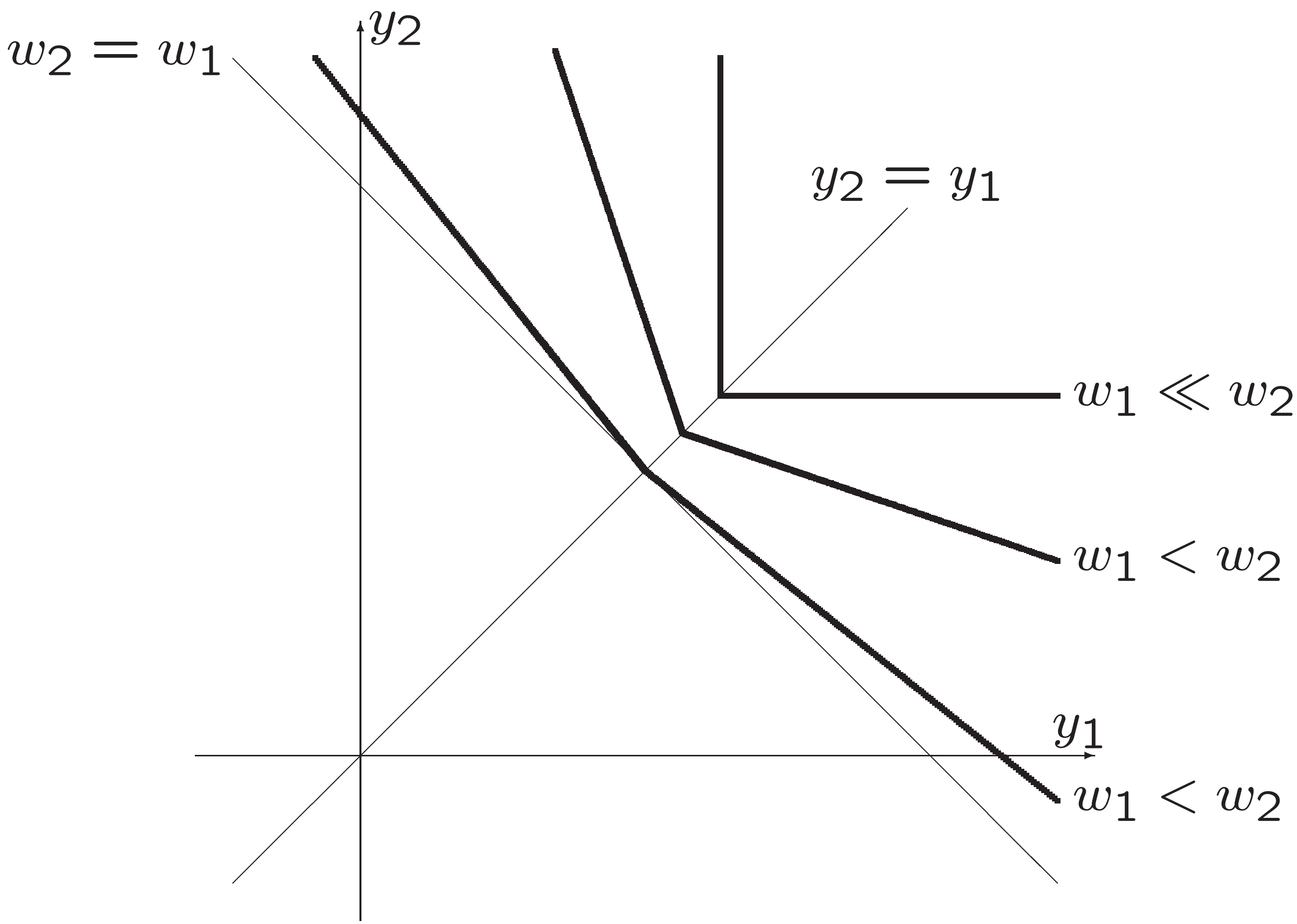
OWA and Risk Aversion

- $0 < w_1 \leq w_2 \leq \ldots \leq w_m$ – totally and-like aggregation
  $w_1 \geq w_2 \geq \ldots \geq w_m > 0$ – totally or-like aggregation
- Decisions under risk
  y_i outcome value under i-th scenario (equally probable)
  $0 < w_1 \leq w_2 \leq \ldots \leq w_m$ – risk aversion for maximized outcome
  $w_1 \geq w_2 \geq \ldots \geq w_m > 0$ – risk aversion for minimized outcome

OWA and Risk Aversion



OWA – monotonic weights



LP computable OWA formula

- $0 < w_1 \leq w_2 \leq \ldots \leq w_m$ – risk averse maximization (and-like)

$$A_{\mathbf{w}}(\mathbf{y}) = \sum_{k=1}^{m} w'_k \bar{\theta}_k(\mathbf{y}) \quad \text{where} \quad \bar{\theta}_k(\mathbf{y}) = \sum_{i \geq k} \theta_i(\mathbf{y}), \quad w'_k > 0$$

$\bar{\theta}_k(\mathbf{y})$ – concave piecewise linear $\Rightarrow$ LP computable [OS, 03]

- LP computability of $\bar{\theta}_k(\mathbf{y})$

$$\bar{\theta}_k(\mathbf{y}) = \min_{u_i} \{ \sum_{i=1}^{m} y_i u_i : \sum_{i=1}^{m} u_i = k, \quad 0 \leq u_i \leq 1 \quad \forall\, i \}$$

by the LP duality

$$\bar{\theta}_k(\mathbf{y}) = \max_{t, d_i} \{kt - \sum_{i=1}^{m} d_i : t - d_i \leq y_i,\ d_i \geq 0 \quad \forall\, i\}$$

Importance Weighted OWA – WOWA

- Preferential weights $\mathbf{w}$: $(\sum_i w_i = 1)$, $w_i \geq 0$
  Importance weights $\mathbf{p}$: $\sum_i p_i = 1$, $p_i \geq 0$
  p_i – measure of ith outcome within distribution of outcomes
  NOT direct rescaling of the ith outcome value

- WOWA aggregation [Torra,97]

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{i=1}^{m} \omega_i \theta_i(\mathbf{y}) \quad \text{with} \quad \omega_i = w^*(\sum_{k \leq i} p_{\tau(k)}) - w^*(\sum_{k < i} p_{\tau(k)})$$

  where w^* is a piecewise linear interpolation of points $(\frac{i}{m}, \sum_{k \leq i} w_k)$ together with (0.0) and τ the ordering permutation for $\mathbf{y}$ (i.e. $y_{\tau(i)} = \theta_i(\mathbf{y})$).

- $p_1 = \ldots = p_m = 1/m \quad \Rightarrow \quad A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = A_{\mathbf{w}}(\mathbf{y})$ (OWA)
  $w_1 = \ldots = w_m = 1/m \quad \Rightarrow \quad A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = A_{\mathbf{p}}(\mathbf{y})$ (Arithmetic Mean)

WOWA orness

- weighting vectors defined via the regular increasing monotone (RIM) quantifiers provide a dimension independent description of OWA
  RIM quantifier Q is (weakly) increasing with $Q(0) = 0$ and $Q(1) = 1$
  The OWA weights defined with Q as $w_i = Q(i/m) - Q((i-1)/m)$

$$\text{orness}(Q) = \int_0^1 Q(\alpha)\, d\alpha$$

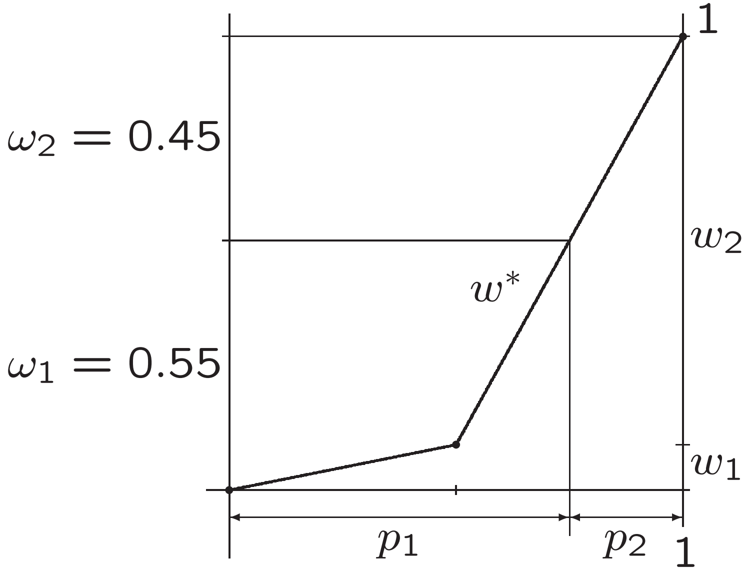
  e.g. orness$(Q) = 1/2$ for $Q(\alpha) = \alpha$ – equal weights $w_k = 1/n$

- WOWA aggregation defined with the RIM quantifier $Q(\alpha) = w^*(\alpha)$
  $0 < w_1 \leq w_2 \leq \ldots \leq w_m$ – convex Q – totally and-like aggregation
  $w_1 \geq w_2 \geq \ldots \geq w_m > 0$ – concave Q – totally or-like aggregation
  in the sense that they remain valid for any subintervals

$$\int_0^1 \frac{Q(a+\alpha(b-a)) - Q(a)}{Q(b) - Q(a)} d\alpha \geq \frac{1}{2} \quad \text{or} \quad \int_0^1 \frac{Q(a+\alpha(b-a)) - Q(a)}{Q(b) - Q(a)} d\alpha \leq \frac{1}{2}$$

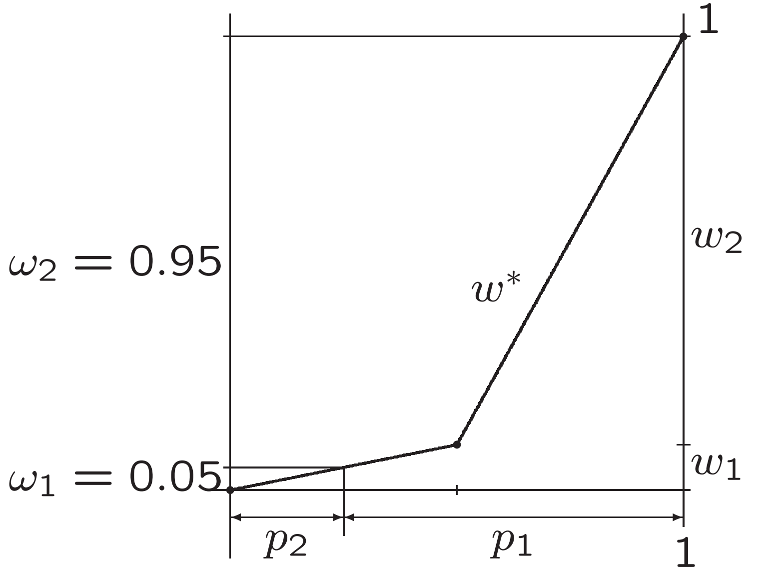
WOWA: $\mathbf{w} = (0.1, 0.9)$, $\mathbf{p} = (0.75, 0.25)$

$a_1 \geq a_2$



$$A_{\mathbf{w},\mathbf{p}}(2,1) = 0.55 \cdot 2 + 0.45 \cdot 1 = 1.55$$

$a_1 \leq a_2$



$$A_{\mathbf{w},\mathbf{p}}(1,2) = 0.05 \cdot 2 + 0.95 \cdot 1 = 1.05$$

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) = \begin{cases} 0.55a_1 + 0.45a_2 & a_1 \geq a_2 \\ 0.05a_2 + 0.95a_1 & a_1 < a_2 \end{cases}$$

WOWA – Reformulation

- OWA weights w_i applied to averages of the corresponding portions of ordered outcomes (quantile intervals) according to the distribution defined by importance weights p_i.

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{i=1}^{m} w_i \left[m \int_{(i-1)/m}^{i/m} \overline{F}_{\mathbf{y}}^{(-1)}(\xi)\, d\xi \right]$$

- Left-continuous right tail cumulative distribution function (cdf):

$$\overline{F}_{\mathbf{y}}(d) = \sum_{i \in I} p_i \delta_i(d) \quad \text{where} \quad \delta_i(d) = \begin{cases} 1 & \text{if } y_i \geq d \\ 0 & \text{otherwise} \end{cases}$$

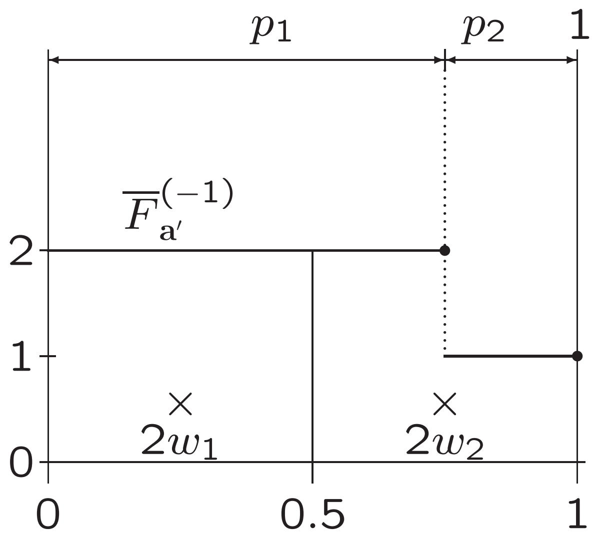
measure of outcomes greater or equal to d.

- Quantile function $\overline{F}_{\mathbf{y}}^{(-1)}$ as the right-continuous inverse of $\overline{F}_{\mathbf{y}}$:

$$\overline{F}_{\mathbf{y}}^{(-1)}(\xi) = \sup \{\eta : \overline{F}_{\mathbf{y}}(\eta) \geq \xi\} \quad \text{for} \quad 0 < \xi \leq 1.$$

WOWA: $\mathbf{w} = (0.1, 0.9)$, $\mathbf{p} = (0.75, 0.25)$

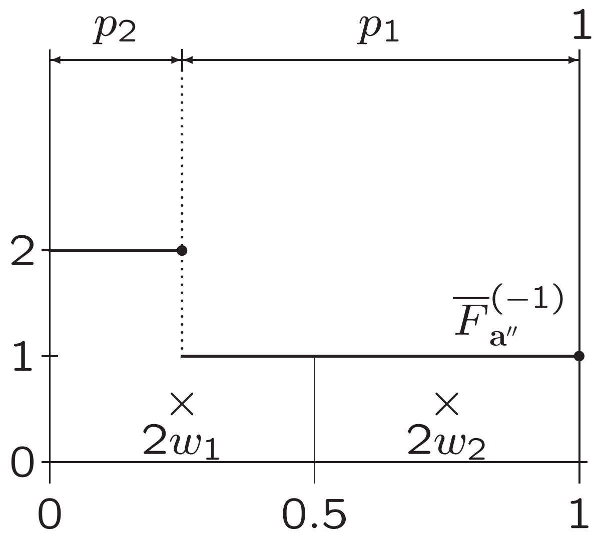
$\mathbf{a}' = (2, 1)$
distribution $(2, 2, 2, 1)$



$$A_{\mathbf{w},\mathbf{p}}(2, 1) =$$
$$0.1 \cdot 2 + 0.9(1 + 2)/2 = 1.55$$

NOT: $(2, 1) \to (1.5, 0.25)$
$0.1 \cdot 1.5 + 0.9 \cdot 0.25 = 0.375$

$\mathbf{a}'' = (1, 2)$
distribution $(2, 1, 1, 1)$



$$A_{\mathbf{w},\mathbf{p}}(1, 2) =$$
$$0.1(1 + 2)/2 + 0.9 \cdot 1 = 1.05$$

NOT: $(1, 2) \to (0.75, 0.5)$
$0.1 \cdot 0.75 + 0.9 \cdot 0.5 = 0.525$

WOWA – LP model [OS, 07]

- Tail averages $L(\mathbf{y},\mathbf{p},\beta)$ (Absolute Lorenz Curve)

$$L(\mathbf{y},\mathbf{p},\beta) = \int_0^\beta F_{\mathbf{y}}^{(-1)}(\alpha)d\alpha = \int_0^\beta \overline{F}_{\mathbf{y}}^{(-1)}(1-\alpha)d\alpha \quad \text{for } 0 < \beta \leq 1$$

- WOWA reformulated

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{k=1}^{m} m w_k (L(\mathbf{y},\mathbf{p},\frac{k}{m}) - L(\mathbf{y},\mathbf{p},\frac{k-1}{m})) = \sum_{k=1}^{m} w'_k L(\mathbf{y},\mathbf{p},\frac{k}{m})$$

risk-averse (monotonic) preferential weights $\Rightarrow \quad w'_k > 0$

- LP computability of ALC

$$L(\mathbf{y},\mathbf{p},\beta) = \min_{u_i} \{ \sum_{i=1}^{m} y_i u_i : \sum_{i=1}^{m} u_i = \beta, \quad 0 \leq u_i \leq p_i \quad \forall\, i \,\}$$

by the LP duality

$$L(\mathbf{y},\mathbf{p},\beta) = \max_{t,d_i} \{\beta t - \sum_{i=1}^{m} p_i d_i : t - d_i \leq y_i,\; d_i \geq 0 \quad \forall\, i\}$$

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